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# q-Laguerre Polynomial Realization of $gl_{\sqrt{q}}(N)$-Covariant Oscillator Algebra

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It is shown that the q-analogue of the group element of $gl_{\sqrt{q}}(N)$-covariant oscillator algebra can be written in terms of the q-deformed Laguerre polynomials.

The theory of quantum groups [1–3] has led to the generalization (deformation) of the oscillator (boson, fermion) algebras in several directions. The development of differential calculus in noncommutative spaces has identified multimode systems of deformed creation and annihilation operators covariant under the action of quantum groups [4–6].

In this paper, we present the realization of the $gl_{\sqrt{q}}(N)$-covariant oscillator algebra and express the q-analogue of the group element of this algebra in terms of the q-deformed Laguerre polynomials.

The $gl_{\sqrt{q}}(N)$ algebra is defined as

$$
\begin{aligned}
a_i a_j &= \frac{1}{\sqrt{q}} a_j a_i \qquad (i < j) \\
a_i^\dagger a_j^\dagger &= \sqrt{q} a_j^\dagger a_i^\dagger \qquad (i < j) \\
a_i a_j^\dagger &= \sqrt{q} a_j^\dagger a_i \qquad (i \neq j) \\
a_i a_1^\dagger - q a_i^\dagger a_i &= 1 + (q - 1) \sum_{k=i+1}^{N} a_k^\dagger a_k
\end{aligned} \tag{1}
$$

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where we restrict our discussion to the case $0 < q < 1$. This algebra is $gl_{\sqrt{q}}(N)$-covariant multimode extension of single oscillator algebra $aa^\dagger - qa^\dagger a = 1$.

The q-deformation of the Bargmann–Fock representation of $gl_{\sqrt{q}}(N)$ oscillators is realized by going over the space of analytic function of n complex variables $z_1, \ldots, z_n$ such as $|z_i|^2 \leq (1-q)^{-1}$, where the operators $a_i$, $a_i^\dagger$, $N_i$ are realized as

$$a_i^\dagger = T_{i+1}^{1/2} \cdots T_n^{1/2} z_i$$

$$a_i = T_{i+1}^{1/2} \cdots T_n^{1/2} \frac{D_i}{1-q}$$

$$N_i = z_i \frac{\partial}{\partial z_i} \tag{2}$$

where

$$D_i = \frac{1 - T_i}{z_i}$$

and

$$T_i f(z_1, \ldots, z_i, \ldots, z_N) = f(z_1, \ldots, qz_i, \ldots, z_N)$$

Let us introduce the polar coordinates $z_i = \rho_i e^{i\theta_i}$. We can define the inner product as

$$\langle f|g\rangle = \left(\frac{1}{2\pi}\right)^N \int_0^{2\pi} \Pi_{i=1}^N d\theta_i \times \int_0^{1/(1-q)} \Pi_{i=1}^N d_q\rho_i^2 \Pi_{i=1}^N E_q(-q(1-q)\rho_i^2)\, \overline{f(z_1, \ldots, z_N)} g(z_1, \ldots, z_N) \tag{3}$$

An orthonormal basis is then given by the product of N monomials

$$|n_1, \ldots, n_N\rangle = \frac{1}{\sqrt{\Pi_{i=1}^N \Gamma_q(n_1 + 1)}} \Pi_{i=1}^N z_i^{n_i} \tag{4}$$

Indeed, one can easily check that $\langle n|m\rangle = \delta_{nm}$.

Using the realization (2), we get the representation

$$N_i|\ldots, n_i, \ldots\rangle = n_i|\ldots, n_i, \ldots\rangle$$

$$a_i|\ldots, n_i, \ldots\rangle = \sqrt{q\Sigma_{k=i+1}^N \frac{1-q^{n_i}}{1-q}}\,|\ldots, n_i - 1, \ldots\rangle$$

$$a_i^\dagger|\ldots, n_i, \ldots\rangle = \sqrt{q\Sigma_{k=i+1}^{N} \frac{1 - q^{n_i+1}}{1 - q}}\,|\ldots, n_i + 1, \ldots\rangle \tag{5}$$

Here, the operator $a_i^\dagger$ is the hermitian conjugate operator of $a_i$.

Now we will discuss the q-analogue of the group element of the $gl_{\sqrt{q}}(N)$-covariant oscillator algebra. For simplicity, we first deal with the $N = 2$ case. In analogy with ordinary Lie theory, we introduce the operator

$$U = U_2(\alpha_2, \beta_2, \gamma_2)U_1(\alpha_1, \beta_1, \gamma_1) \tag{6}$$

where

$$U_i(\alpha_i, \beta_i, \gamma_i) = E_q(\alpha_i(1 - q)a_i^\dagger)E_q(\beta_i(1 - q)a_i)E_q(\gamma_i(1 - q)N_i) \tag{7}$$

and the q-exponential function $E_q(x)$ is defined by

$$E_q(x) = \sum_{n=0}^{\infty} \frac{q^{n(n-1)/2}}{(q; q)_n} x^n$$

Here, the q-shifted factorial is defined by

$$(a; q)_\infty = \prod_{k=0}^{\infty} (1 - aq^k)$$

$$(a; q)_n = \frac{(a; q)_\infty}{(aq^n; q)_\infty}$$

We then define the matrix element $U_{kl}^{nm}$ through

$$Uz_1^n z_2^m = \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} U_{kl}^{nm} z_i^k z_2^l \tag{8}$$

After some calculation, we have

$$\begin{aligned} U_{kl}^{nm} = {} & E_q(\gamma_2(1 - q)m)E_q(\gamma_1(1 - q)n)q^{\frac{1}{2}(n-k)(n+m-k-1)} q^{\frac{1}{2}(m-l)(m-l-1)} \beta_1^{n-k}\beta_2^{m-l} \\ & \times L_k^{n-k}(-q^{m-1}\alpha_1\beta_1; q)L_l^{m-l}(-q^{-1}\alpha_2\beta_2; q) \end{aligned}$$

where the q-Laguerre function is defined by

$$L_k^\lambda = \sum_{l=0}^{k} \frac{(q; q)_{k+\lambda} q^{l(l+\lambda)}(1 - q)^l}{(q; q)_{k-l}(q; q)_{l+\lambda}(q; q)_l} (-x)^l \tag{10}$$

From $U_2(-\alpha_2, q, 0)U_1(-\alpha_1, q, 0)$, we get the identity

$$E_q(-\alpha_2(1-q)z_2)E_q(-\alpha_1(1-q)q^{m/2}z_1)\left(-\frac{q}{z_2};q\right)_m\left(-\frac{q}{z_1}q^{m/2};q\right)_n$$
$$=\sum_{k,l=0}^{\infty} q^{\frac{1}{2}(n-k)(n+m-k+1)} q^{\frac{1}{2}(m-l)(m-l+1)} L_k^{n-k}(q^m\alpha_1;q)L_l^{m-l}(\alpha_2)$$

Our formula can be easily extended to the general $N$ case. Then, the q-analogue of the group element is defined by

$$U(\alpha,\beta,\gamma)\Pi_{j=1}^{N}z_j^{n_j}$$
$$=U_N(\alpha_N,\beta_N,\gamma_N)\cdots U_1(\alpha_1,\beta_1,\gamma_1)\Pi_{j=1}^{N}z_j^{n_j}$$
$$=\sum_{k_1,\ldots,k_N=0}^{\infty} U_{k_1,\ldots,k_N}^{n_1,\ldots,n_N}\Pi_{j=1}^{N}z_j^{n_j} \tag{11}$$

Then, the matrix element $U_{k_1,\ldots,k_N}^{n_1,\ldots,n_N}$ is computed as

$$U_{k_1,\ldots,k_N}^{n_1,\ldots,n_N}=\Pi_{i=1}^{N}[E_q(\gamma_i(1-q)N_i)q^{\frac{1}{2}(n_i-k_i)(\Sigma_{k=1}^{N}n_k-k_i-1)}$$
$$\times\beta_i^{n_i-k_i}L_{k_i}^{n_i-k_i}(-q^{\Sigma_{k=i+1}^{N}n_k-1}\alpha_i\beta_i;q)]. \tag{12}$$

To conclude, in this paper, I have discussed the q-special function realization of the q-analogue of the group element for the $gl_{\sqrt{q}}(N)$-covariant oscillator algebra. I found that the q-analogue of the group element is written in terms of the product of the q-Laguerre polynomials.

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